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Approximation Properties of Beta Operators

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A localization theorem for Beta approximation operators β_n $(n = 1, 2,...)$, $\beta_n(f; x) = \int_0^\infty b_n(x, u) f(u^{-1})\, du$, where

$$b_n(x, u) = (x^n/B(n, n))(u^{n-1}/(1 + xu)^{2n}), \qquad x > 0$$

has been proved and with the help of this theorem the uniform convergence of $\beta_n f$ to f every fixed interval $[x_1, x_2]$ $(0 < x_1 \leqslant x_2 < \infty)$ has been established.

1. INTRODUCTION

In [3, Chap. VI], while studying conditions for the regularity of sequence-to-sequence transformations, beta transform arose naturally. The beta transform of order (p, q) is defined as

$$\mathscr{B}_{pq}[\phi(t); x] = \int_0^\infty \frac{t^{q-1}}{(1 + xt)^{p+q}} \phi(t)\, dt \qquad (\mathrm{Re}\, p > 0, \mathrm{Re}\, q > 0, \mathrm{Re}\, x > 0). \tag{1.1}$$

It has been discussed briefly in [3, Chap. VII].

Using the beta transform kernel, $t^{n-1}/(1 + xt)^{m+n}$ $(m, n \geqslant 1)$, a double sequence of linear, positive, integral operators β_{mn} $(m, n = 1, 2,...)$ has been introduced in [3, Chap. IX]. The (m, n)th beta operator is

$$\beta_{mn}(f; x) = \int_0^\infty b_{mn}(x, u) f(n/mu)\, du, \tag{1.2}$$

where

$$b_{mn}(x, u) = \frac{x^n}{B(m, n)} \frac{u^{n-1}}{(1 + xu)^{m+n}} \qquad (m, n = 1, 2,...),$$

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$x>0$ and $f(\cdot)\in M[0,\infty)$ ($M[0,\infty)$ is the linear space of the functions $f(t)$ defined for $t\geqslant 0$ and bounded and Lebesgue-measurable in every interval $[r, R]$ $(0<r<R<\infty)$ [1, Definition 1.1]).

Some elementary properties and estimates for these operators have been given in [5] and [6]. It has been proved in [5, Theorem 2.1] that

$$\lim_{m\to\infty,\, n\to\infty} \beta_{mn}(f;x)=f(x), \qquad \text{uniformly in } [x_1, x_2], \tag{1.3}$$

if $f(\cdot)\in S[x_1,x_2]$ ($S[x_1,x_2]$ is the linear space of bounded functions $f(\cdot)\in M[0,\infty)$, continuous at all points of the fixed interval $[x_1,x_2]$ $(0\leqslant x_1<x_2<\infty)$). In case $x_1=0$, the continuity at x_1 is one-sided [1, Definition 2.1].

To avoid the double limit, we take $m=n$ and obtain a sequence of the operators

$$\beta_n(f;x)=\int_0^\infty b_n(x,u)f(u^{-1})\,du \qquad (x>0), \tag{1.4}$$

where

$$b_n(x,u)=\frac{x^n}{B(n,n)}\frac{u^{n-1}}{(1+xu)^{2n}} \qquad (n=1,2,...),\ f(\cdot)\in M[0,\infty).$$

It also follows easily that if $f(\cdot)\in S[x_1,x_2]$, then

$$\lim_{n\to\infty} \beta_n(f;x)=f(x) \qquad \text{uniformly in } [x_1,x_2]. \tag{1.5}$$

Lupaş [2] has also introduced a sequence of linear, positive, integral operators $\mathbb{B}_n$ (termed beta operators) as follows:

$$(\mathbb{B}_n f)(x)=\int_0^1 \beta_n(t,x)f(t)\,dt \qquad (n=1,2,...), \tag{1.6}$$

where

$$\beta_n(t,x)=\frac{1}{B(nx+1,n+1-nx)}\,t^{nx}(1-t)^{n(1-x)}, \qquad x\in[0,1].$$

The kernel of these operators is from the beta distribution with positive parameters $\bar p, \bar q$ and with the probability density function

$$\begin{aligned} b_{p,q}(t)&=0, && -\infty<t\leqslant 0;\\ &=t^{\bar p-1}(1-t)^{\bar q-1}/B(\bar p,\bar q), && 0<t<1;\\ &=0, && 1\leqslant t<\infty;\end{aligned}$$

$$\bar p=nx+1, \qquad \bar q=n+1-nx.$$

There is quite a difference between the definition and properties of the operators (1.4) and (1.6) but both are closely related to the beta distribution of probability theory.

In this paper we give a localization theorem and a convergence theorem (based on the localization theorem) for the operators (1.4).

2. The Results

Definition 2.1 (Functional Space $H(0, \infty)$). $H(0, \infty)$ is the linear space of the functions $f(x) \in M[0, \infty)$ for which $|f(x)| \leqslant Px^{\alpha}$ ($P>0$, $\alpha>0$, $x>0$).

Lemma 2.1. *If $f(x) \in H(0, \infty)$, then $\beta_n(f; x)$ exists for all $n \geqslant [\alpha]+1$.*

Theorem 2.1. *Let $f(x)$ and $g(x)$ be functions such that*

(i) $f(x) \in H(0, \infty)$,

(ii) $g(x) \in M(0, \infty)$,

(iii) *$f(x)$ is continuous adnd $=$ $g(x)$ at every point of the fixed interval $[x_1, x_2]$ $(0<x_1 \leqslant x_2<\infty)$.*

Then $\beta_n(f; x)$ exists for $x \geqslant x_1$, $n \geqslant [\alpha]+1$, and

$$\lim_{n \to \infty} \beta_n(f; x) = \lim_{n \to \infty} \beta_n(g; x) \tag{2.1}$$

in $[x_1, x_2]$, the convergence holding there uniformly.

Proof. Let $n \geqslant [\alpha]+1$. Then by Lemma 2.1, $\beta_n(f; x)$ exists for $x \in [x_1, x_2]$ $(0<x_1 \leqslant x_2<\infty)$. Hypothesis (iii) implies the existence of a number $\delta=\delta(\varepsilon)>0$, independent of $x \in [x_1, x_2]$, and such that

$$|f(u^{-1}-f(x)|<\varepsilon/2 \qquad \text{and} \qquad |g(u^{-1})-g(x)|<\varepsilon/2 \tag{2.2}$$

for $|u^{-1}-x|<\delta$, $n>0$, $u>0$, and $x \in [x_1, x_2]$. Also, by Hypotheses (i) and (ii), we have

$$|f(u^{-1})-g(u^{-1})| \leqslant Pu^{-\alpha}+M, \tag{2.3}$$

where $M=\sup_{0<t<\infty} |g(t)|$. Now, for a fixed $x \in [x_1, x_2]$, we have

$$\begin{aligned} \beta_n(f; x)-\beta_n(g; x) &= \int_0^{\infty} b_n(x, u)[f(u^{-1})-g(u^{-1})]\, du \\ &= J_n^1+J_n^2, \end{aligned}$$

where

$$J_n^i = \int_{u \in N_i} b_n(x, u)[f(u^{-1}) - g(u^{-1})]\, du \qquad (i = 1, 2),$$

$$N_1 = \left\{u\colon u > 0,\ |u - x^{-1}| < \frac{\delta}{x(x+\delta)}\right\} \qquad \text{and} \qquad N_2 = [0, \infty) - N_1.$$

With the help of (2.2) and (2.5) of [1], and Hypothesis (iii) we obtain

$$|J_n^1| \leqslant \varepsilon. \tag{2.4}$$

Also, by (2.3) we have

$$\begin{aligned}
|J_n^2| &\leqslant M \int_{u \in N_2} b_n(x, u)\, du + P \int_{u \in N_2} b_n(x, u)\, u^{-\alpha}\, du \\
&\leqslant \frac{M}{t^2} \int_0^{\infty} b_n(x, u)(u - x^{-1})^2\, du \\
&\quad + \frac{P}{t^2} \int_0^{\infty} b_n(x, u)\, u^{-\alpha}(u - x^{-1})^2\, du \left(t = \frac{\delta}{x(x+\delta)}\right) \\
&\leqslant \left(\frac{x_2 + \delta}{\delta}\right)^2 \left[M \frac{2(n+1)}{(n-1)(n-2)} \right. \\
&\quad + P x_2^{\alpha} \left\{ \frac{\Gamma(n - \alpha + 2)}{\Gamma(n)} \frac{\Gamma(n + \alpha - 2)}{\Gamma(n)} \right. \\
&\quad \left.\left. - 2 \frac{\Gamma(n - \alpha + 1)}{\Gamma(n)} \frac{\Gamma(n + \alpha - 1)}{\Gamma(n)} + \frac{\Gamma(n - \alpha)}{\Gamma(n)} \frac{\Gamma(n + \alpha)}{\Gamma(n)} \right\} \right].
\end{aligned}$$

We may choose a number n_ε, sufficiently large and such that

$$|J_n^2| \leqslant \varepsilon \qquad \text{for} \quad n > n_\varepsilon. \tag{2.5}$$

(It is clear that n_ε is independent of $x \in [x_1, x_2]$.) Thus, from (2.4) and (2.5) we have

$$|\beta_n(f; x) - \beta_n(g; x)| \leqslant 2\varepsilon \qquad (n > n_\varepsilon),$$

for every $x \in [x_1, x_2]$. This proves the theorem.

The following theorem is an immediate consequence of the preceding one.

THEOREM 2.2. *Let $f(x) \in H(0, \infty)$ be continuous at all points of the interval $[x_1, x_2]$ $(0 < x_1 \leqslant x_2 < \infty)$. Then $\beta_n(f; x)$ exists for $x \geqslant x_1$, $n \geqslant [\alpha] + 1$, and*

$$\lim_{n \to \infty} \beta_n(f; x) = f(x), \qquad \textit{uniformly in } [x_1, x_2]. \tag{2.6}$$

Proof. The first part of the theorem follows by Lemma 2.1. Let

$$g(x)=f(x) \qquad \text{in } [x_1, x_2].$$

The precise values of $g(x)$ at the remaining points of the interval $(0, \infty)$ (i.e., for $0<x<x_1$ and $x_2<x<\infty$) are unimportant, but we assume that g is bounded and Lebesgue-measurable there.

Both functions $f(x)$ and $g(x)$ satisfy the assumptions made in Theorem 2.1, thereby giving

$$\lim_{n\to\infty} \beta_n(f;x)=\lim_{n\to\infty} \beta_n(g;x) \qquad \text{in } [x_1, x_2], \text{ the convergence holding there uniformly.}$$

Also, by [5, Theorem 2.1] we have

$$\lim_{n\to\infty} \beta_n(g;x)=g(x), \qquad \text{uniformly in } [x_1, x_2].$$

Summing up these results, we have

$$\lim_{n\to\infty} \beta_n(f;x)=g(x), \qquad \text{uniformly in } [x_1, x_2].$$

Since $f(x)=g(x)$ in $[x_1, x_2]$, the proof of the theorem follows.

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